Shape Factor (s)

Let $s=\frac{b}{r}$ as $0 \ll b \leqslant 2r$ then $0 \ll s \ll 2$ 1

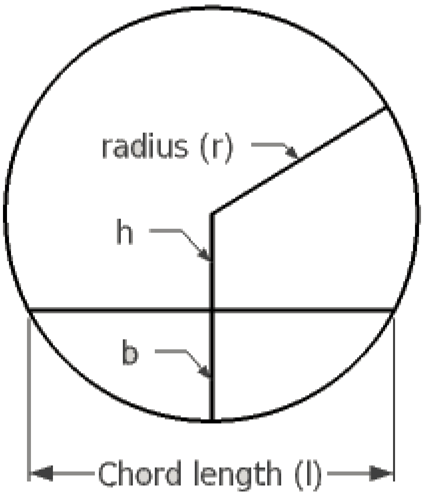


Figure 1: Circle with chord

The chord equation is

$$r=\frac{(b^2+(l/2)^2)}{2b}$$ 2

from eq. 1 b = sr substituting in eq 2 gives

$$r=\frac{(s^2r^2+(\frac{l}{2})^2)}{2sr}$$ 3

re-arranging eq 3 gives

$$r=\frac{(l/2)}{\sqrt{s(2-s)}}$$ 4

From the diagram b = r – h substituting in eq 1 gives

$$s=\frac{r-h}{r}$$ 5

re-arranging for h (distance from the centre of the circle to the chord)

$$h=r(1-s)$$ 6

Equations 4 and 6 will be used to determine the radii of the top and bottom of the truncated cone and the distance (h) that the chords are from the axis of the cone. These chords (which are the inner (l_i**)** and outer (l_o) segment lengths) are invariant to changes in the segment width (w_i) and the shape factors (s_i and s_o) and the “dish” of the ring.

Practical limits for the shape factor (s)

Figure 2 shows an end view of three cones assembled as they would be in a ring, such that the outside walls will just touch. This is a limiting condition for the shape factor (s).

The true value of the wall thickness in this perspective is $wt/\cos(\beta)$ however as β (cone half angle) is reasonably small and we are going to develop an approximate set of limits, the wall thickness will be approximated by wt

Consider the right angle triangle connecting the centres of the top and right hand cones. It has an hypotenuse of 2*r, vertical side 2*h and the horizontal side of r+wt/2. By similar triangles the small triangle has a horizontal side of:

$$l/2=\frac{(r+wt/2)}{2} \qquad 7$$

substituting eq. 4 into 7 gives

$$l=\frac{l}{2\sqrt{s(2-s)}}+wt/2 \qquad 8$$

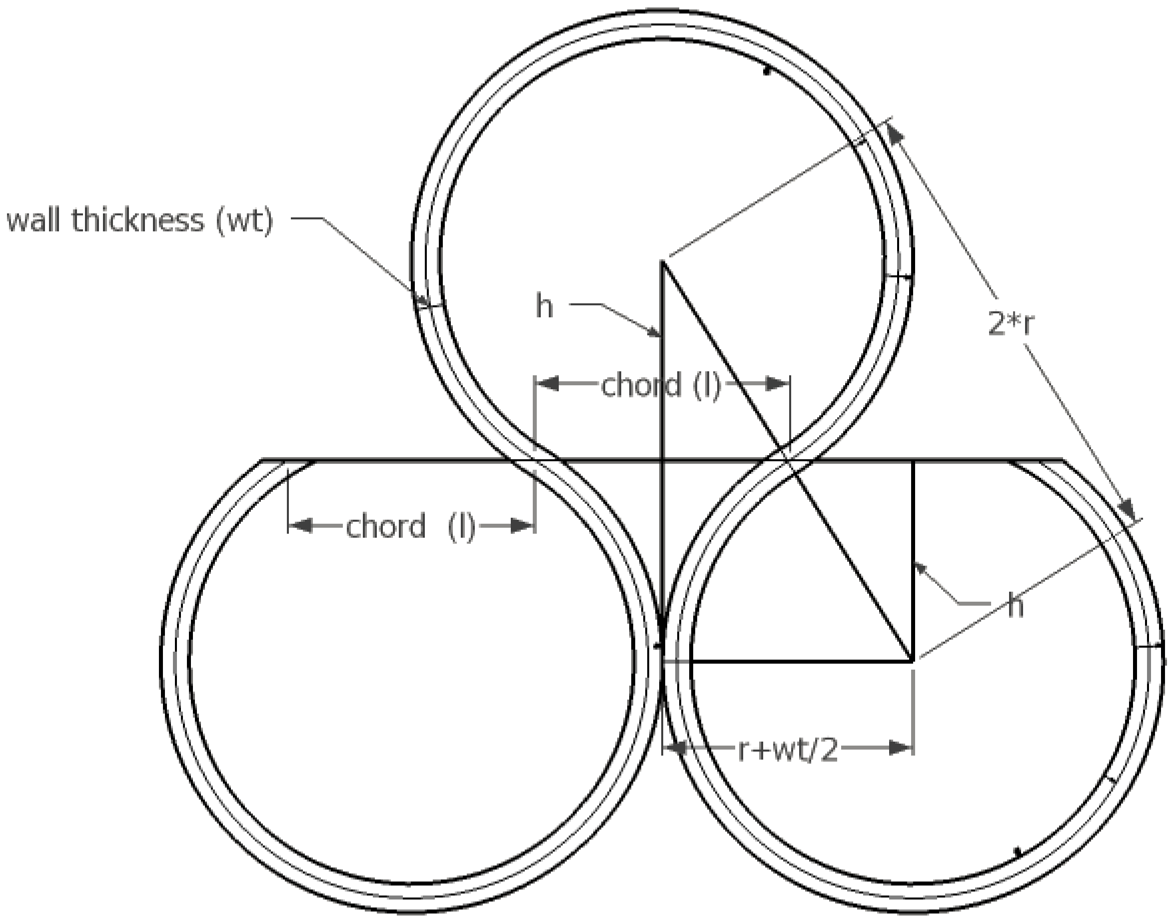


FIGURE 2: Extreme limit - walls touching

Transposing the last term on the RHS and squaring both sides gives

$$\frac{l^2}{4s(2-s)}=(l-wt/2)^2 \qquad 9$$

Which can be re-arranged to be

$$s^2-2s+\frac{1}{(2-wt/l)^2}=0 \qquad 10$$

This quadratic equation can be solved as $s=1\pm\sqrt{1-\frac{1}{(2-wt/l)^2}}$ 11

This gives the upper limit for s. Note that this limit is a function of the ratio of the wall thickness and segment inner or outer lengths. We’ll analyse this equation later and determine some approximate fixed limits. First, there is another condition which will lead to limits on s, that is when the chord becomes tangential to the inner wall. Figure 3 illustrates this condition.

Equating eqs. 2 and 4 and substituting wt/2 for b gives

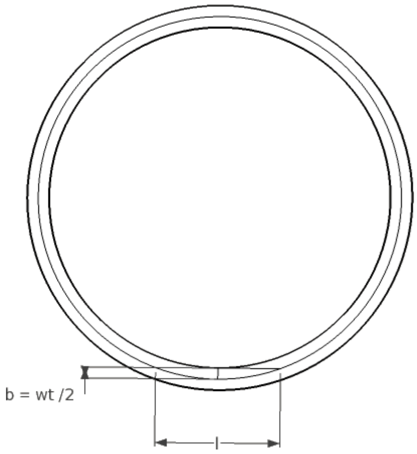


FIGURE 3: Chord tangential to the inner wall

$$\frac{l/2}{\sqrt{s(2-s)}}=\frac{(wt/2)^2+(l/2)^2}{wt} \quad 12$$

re-arranging gives

$$\sqrt{s(2-s)}=\frac{2}{wt/l+l/wt} \quad 13$$

squaring both sides

$$s(2-s)=\frac{4}{(wt/l+l/wt)^2} \quad 14$$

re-arranging gives

$$s^2-2s+\frac{4}{(wt/l+l/wt)^2}=0 \quad 15$$

Solving this quadratic gives

$$s=\frac{2\pm\sqrt{4-\frac{16}{(wt/l+l/wt)^2}}}{2} \quad 16$$

which simplifies to

$$s=1\pm\frac{\sqrt{(wt/l+l/wt)^2-4}}{wt/l+l/wt} \quad 17$$

now

$$(wt/l+l/wt)^2-4=(wt/l-l/wt)^2 \quad 18$$

substituting eq 18 into eq 17 and simplifying gives

$$s=1\pm\frac{(wt/l)^2-1}{(wt/l)^2+1} \quad 19$$

taking the + case and re-arranging gives

$$s=\frac{2wt/l}{(wt/l)^2+1} \quad 20$$

This equation 20 determines the lower value for s.

A reasonable range for wt/l is 0.1 (large cones) to 0.4 (for small cones). Graphing eq.11 and eq. 20 over this range of wt/l gives figure 4. .

the cone wall thickness (wt) will be much less than the inner length of the segment (li) and it is unlikely to be more than 30% of that length. We can see from the graph of figure 4 that up to a ratio $wt/l=0.33$ equation 11 gives s = 1.8. This ratio would give rise to wall thicknesses greater than is needed right down to a segment length of 10mm (3/8in). At this ratio equation 20 gives a value of s of approximately 0.3. It is reasonable then to choose limits for the shape factors to be between 0.3 and 1.8

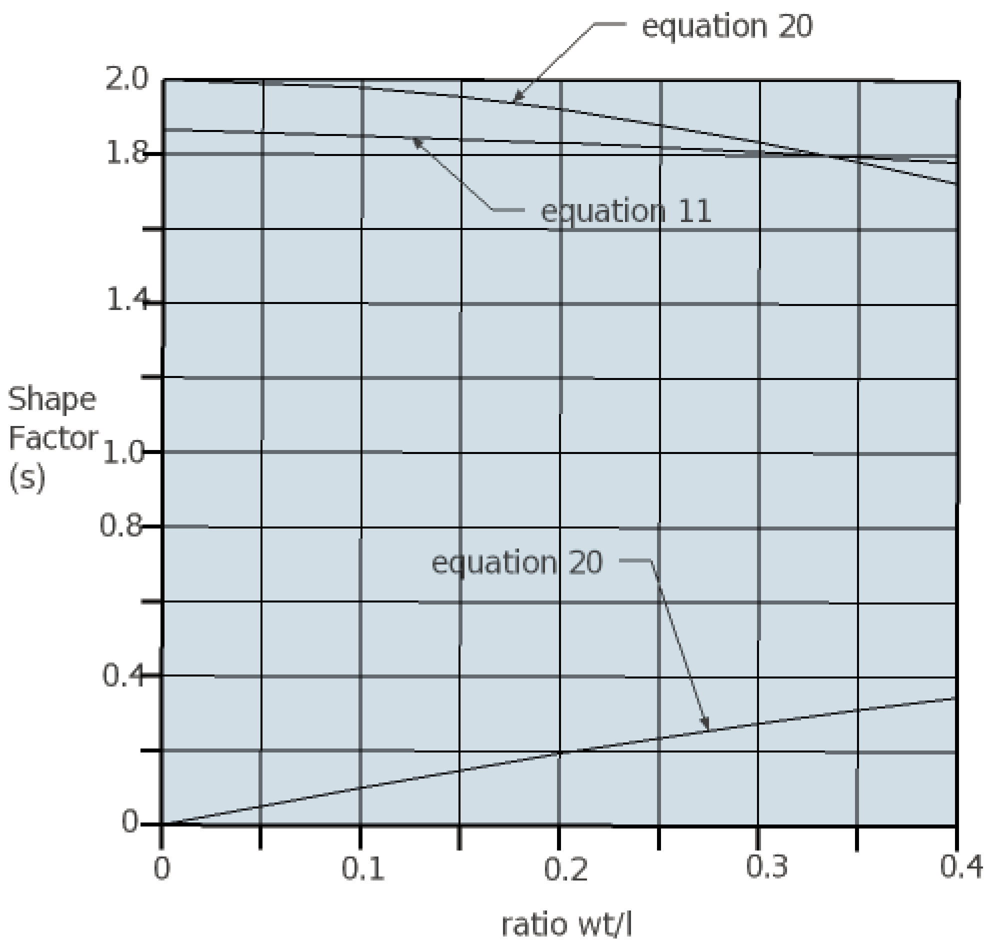


Figure 4   Shape factor limits

Cone dimension analysis

These are the variables which will be used or generated in the development :-

- ro the ring's outer radius.
- ri the ring's inner radius.
- lo the segments outer length
- li the segments inner length
- n the number of segments in the ring
- Θ half of the segments subtended angle

wi the width of the segment

wt cone wall thickness

rl radius of the cone's large end

rs radius of the cones small end

so shape factor for the cone's large end

si shape factor for the cone's small end

wj length of the joint surface

wjx extended length of the joint surface

wl length of the work pieces

wtu upper work piece thickness

wtl lower work piece thickness

ww width of the work pieces

ol offset from joint to the cone axis at large end of cone

os offset from joint to the cone axis at small end of cone

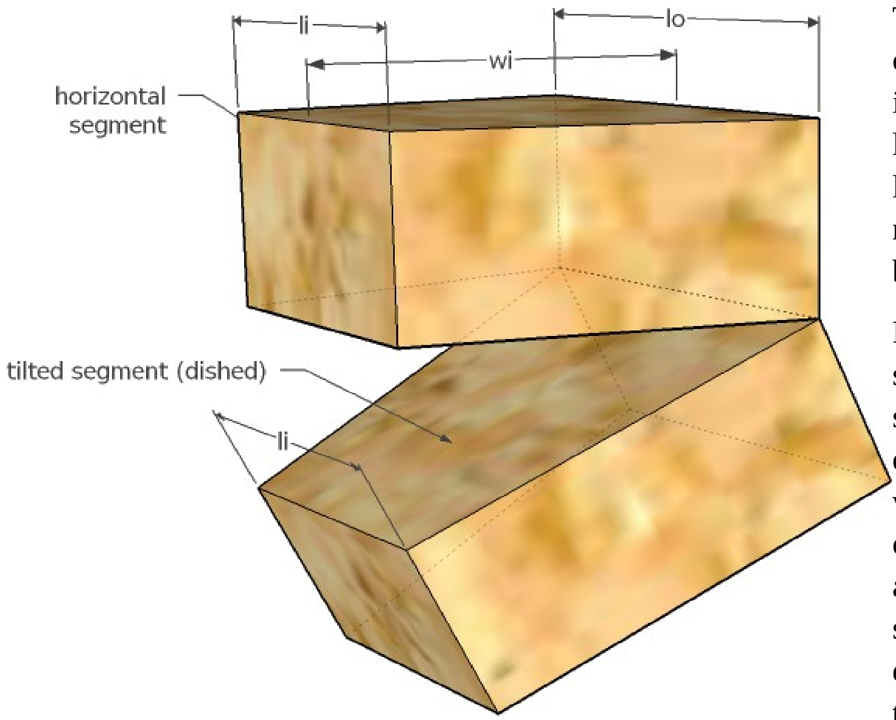
α angle between the joint and the cone axis

β angle between the cone outer edge and it's axis

δ angle between the joint and the horizontal plane of the assembled ring

The segment details (lo,li,wi and Θ) are calculated in the usual way.

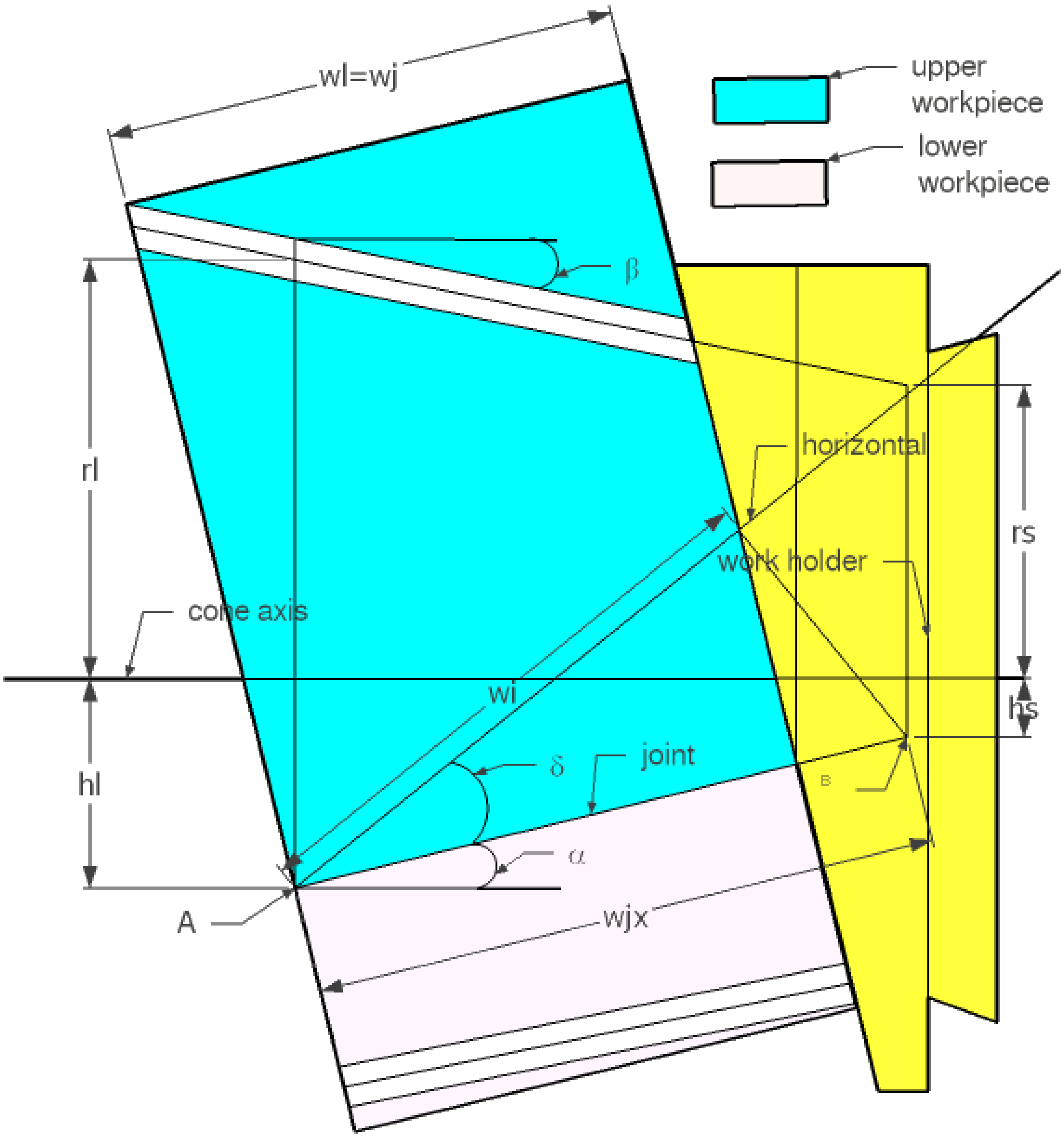
Diagram 5 Comparison of horizontal and tilted segments



The circles at the ends of the cone are at right angles to it's axis and the chords that lie across them are exactly lo and li and these chords must in the assembled ring be wi horizontally apart.

Figure 5 illustrates this for a simple segment. The top segment in the diagram if duplicated and assembled would produce a ring with outside and inside radii of ro and ri. If the bottom segment were also duplicated and assembled the ring produced would also have those same radii, both rings would be equivalent but one would be "dished". Note that the lengths of the segment (lo) and (li)are preserved during the tilt as is the width of the assembled ring but that the segment width (wi) is not. and that the inner and outer surfaces of the ring will follow the tilt of the segment. This latter is not a necessity but is does render the work pieces rectangular and simplifies mounting them.

Figure 6 cross section of work pieces, holder and subsequent cone



The diagram we saw in the design implementation page has been extended here, (Figure 6) to show some further details important to this analysis, in particular the line marked “horizontal” which will when the pieces of the cones are assembled be in the horizontal plane. In order to better understand the diagram in figure 6 we’ll build up the diagram line by line.

At right angles to the joint line are 2 lines (wj) apart which will be the edges of the outer and inner surfaces of the segment. Next we have the line marked “horizontal” which will be horizontal when the ring has been assembled. The angle separating these lines (δ) is derived from the “dish” and ring outer radius. Now we can put in the cone axis (which is shown horizontal on the page as this is the view you will see when the assembly is on the lathe).

The 2 narrow parallelograms are cross sections of the final cone and are the edges of the cone's inner and outer surfaces, of course they are symmetric about the cone axis Then there are 2 lines (top and bottom) parallel to the joint line and touching one of the points of the narrow parallelograms. These lines, the lines at right angles to the joint line and the joint line are the outlines of the 2 work pieces. Finally on the right hand side we have an outline of the work holder suitable for mounting in a chuck.

The points A and B on figure 6 are where the chords lo and li lie (directly into the paper) and these are the outer and inner lengths of the segment respectively The vertical lines (rl,hl,rs,hs) are the edges of the planes on which our cone ends are defined. Assuming that the ring details, shape factors (so,si) and the amount of "dish" are known then we can proceed.

Using eq. 4 we have

$$rl=\frac{lo/2}{\sqrt{so(2-so)}} \quad 21$$

$$rs=\frac{li/2}{\sqrt{si(2-si)}} \quad 22$$

using eq. 6 we have

$$hl=rl(1-so) \quad 23$$

$$hs=rs(1-si) \quad 24$$

and

$$\delta=\arctan\left(\frac{dish}{ro}\right) \quad 25$$

now

$$wjx=\frac{wi}{\cos(\delta)} \quad 26$$

from figure 6 we have

$$\alpha=\arcsin\left(\frac{hs-hl}{wjx}\right) \quad 27$$

now we have α we can determine β as

$$\beta=\arctan\left(\frac{rs-rl}{wjx\cos(\alpha)}\right) \quad 28$$

We now have all the lengths and angles needed to determine dimensions of the upper and lower work pieces. Before we do we should look at those situations where the "dish" or the shape factor for the small end of the cone (si) are unknown. These are where one surface of the ring is to be horizontal and both to be horizontal, respectively.

One surface to be horizontal

This will be the top surface of the cone cross section on the diagram of figure 6. During assembly the ring will be turned over and this will become the bottom. Here the amount of dish will be used to adjust the orientation of the surface thus the “dish” is unknown a priori and δ cannot be directly calculated.

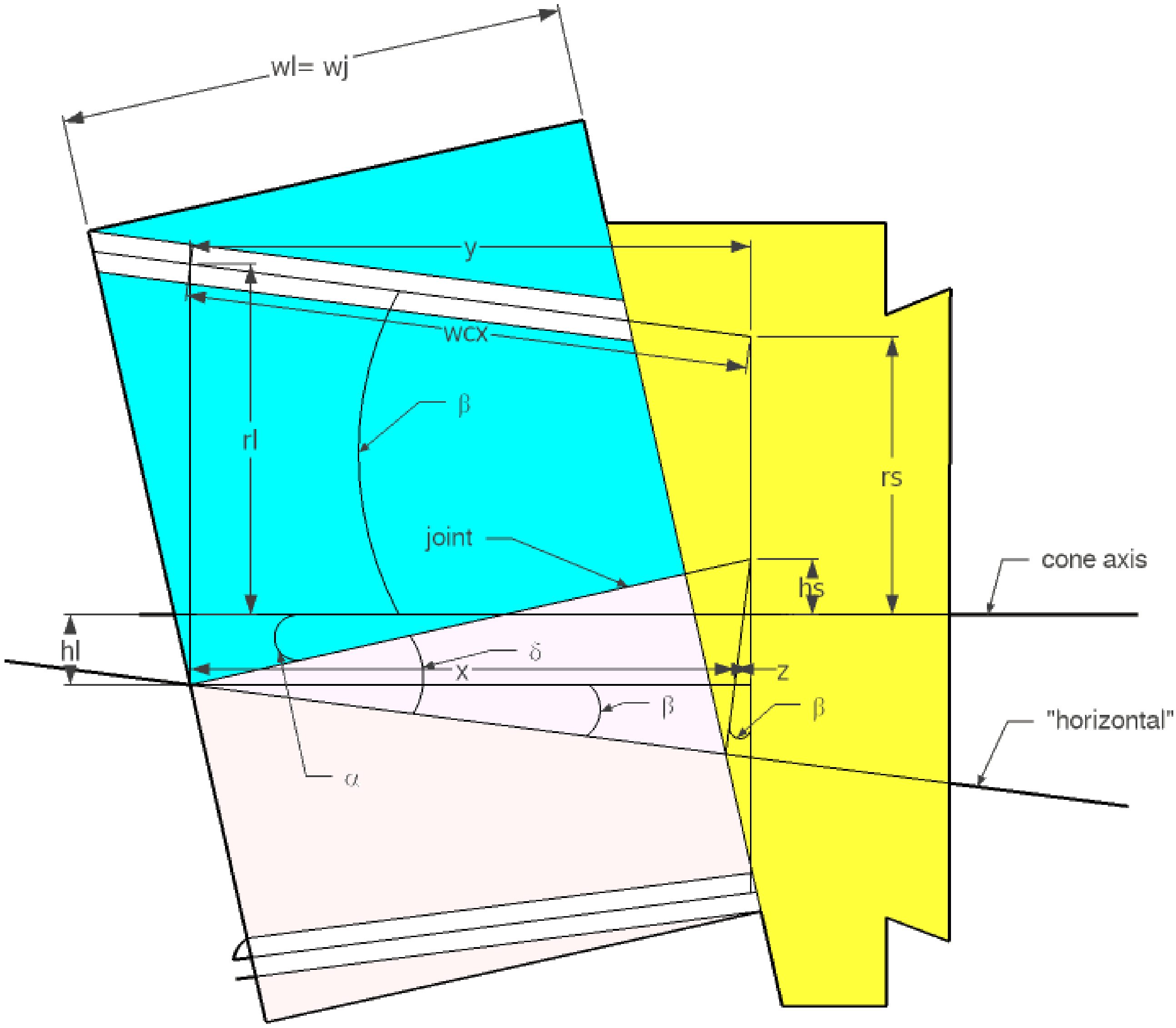


diagram 6A cone cross section for a ring with one surface horizontal

From diagram 6a we can see that the line of length y is the addition of the lines x and z. From this we get

$$y=\frac{rs-rl}{\tan(\beta)}=\left(\frac{wi}{\cos(\beta)}=x\right)-\left((hs-hl)\tan(\beta)=z\right) \quad 29$$

hence

$$\frac{(rs-rl)\cos(\beta)}{\sin(\beta)}=\frac{wi-(hs-hl)\sin(\beta)}{\cos(\beta)} \quad 30$$

multiplying both sides by sin(β)cos(β) gives

$$(rs-rl)\cos^2(\beta)=wi\sin(\beta)-(hs-hl)\sin^2(\beta) \quad 31$$

noting that $\cos^2(\beta)=(1-\sin^2(\beta))$ gives

$$((rs-rl)-(hs-hl))\sin^2(\beta)+wi*\sin(\beta)-(rs-rl)=0 \tag{32}$$

this quadratic solves to give

$$\sin(\beta)=\frac{-wi\pm\sqrt{wi^2+4((rs-rl)-(hs-hl))(rs-rl)}}{2((rs-rl)-(hs-hl))} \tag{33}$$

We now need α

$$\sin(\alpha)=\frac{(hs-hl)\cos(\delta)}{wi} \tag{34}$$

but

$$\delta=\alpha-\beta \tag{35}$$

$$\sin(\alpha)=(\frac{hs-hl}{wi})\cos(\alpha-\beta) \tag{36}$$

using the difference of angles formula $\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$ gives

$$\sin(\alpha)=(\frac{(hs-hl)}{wi})(\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)) \tag{37}$$

Re-arranging gives

$$\tan(\alpha)=\frac{\cos(\beta)}{\frac{wi}{hs-hl}-\sin(\beta)} \tag{38}$$

The equations 33,35 and 38 are used in place of eqs 25,27 and 28.

Both surfaces of the ring to be horizontal

If both surfaces of the ring are to be horizontal then it follows that the joint surface must also be horizontal and so “dish” must be 0 and we will need to adjust one of the shape factors. For the top surface of the cone to be parallel to the joint surface it follows that

$$(rl-hl)=(rs-hs) \tag{39}$$

using eqs 4 and 6 we can substitute for rs and hs to get

$$(rl-hl)=\frac{(li/2)si}{\sqrt{si(2-si)}} \tag{40}$$

solving this for si by squaring both sides and re-arranging we get

$$si=\frac{2(rl-hl)^2}{(li/2)^2+(rl-hl)^2} \tag{41}$$

We may now use eqs 22 and 24 to obtain rs and hs. And then eqs 25-28

Extending the chords

The outer and inner lengths of the segment (lo and li) need to be extended to the outside surface of the cone as these are the lengths (chords) that can be measured during turning.

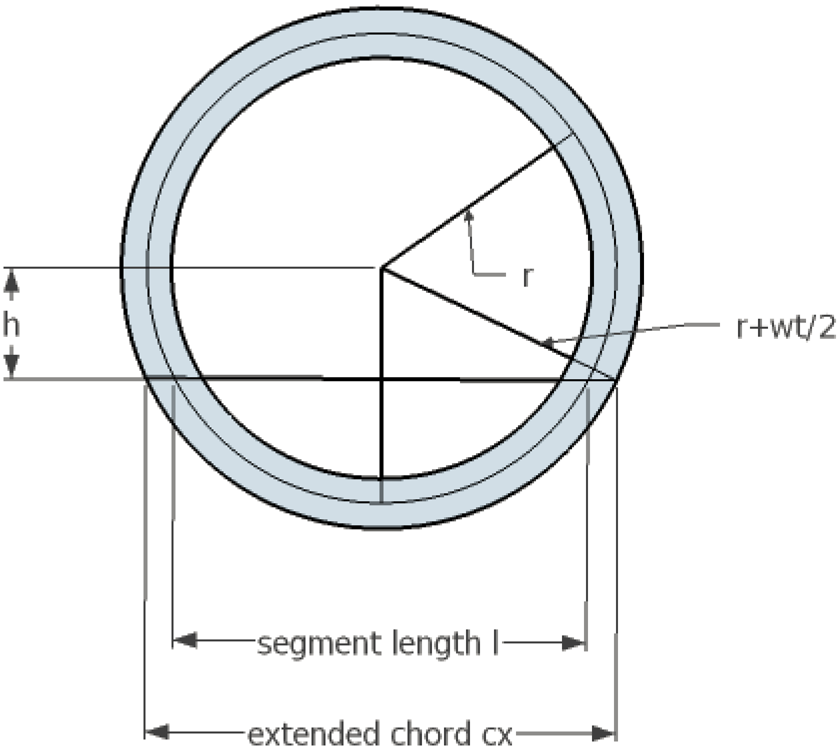


Figure 7 Extending the chord

From figure 7 we get that

$$cx = 2\sqrt{(r + wt/(2\cos(\beta)))^2 - h^2}$$

Therefore

$$cxl = 2\sqrt{(rl + wt/(2\cos(\beta)))^2 - hl^2} \quad 42$$

$$cxs = 2\sqrt{(rs + wt/(2\cos(\beta)))^2 - hs^2} \quad 43$$

Dimensioning the work pieces

From figure 6 we can see that the length of the work pieces (wl) is the same as the length of the joint

$$wl = wj \quad 44$$

width of the work pieces is

$$ww = cxl \quad 45$$

The work pieces are cuboids except for the symmetric case when their end angles = (90 – δ)

The distance from the joint line to the cone axis along the work piece end is

$$ol = \frac{hl}{\cos(\alpha)} \quad 46$$

$$os = \frac{hs}{\cos(\alpha)}$$

We are left now with determining the thickness of each work piece. Looking at figure 6 we have two situations. The first is when the edge of the work piece just touches the cone outside surface (upper work piece top edge in fig 6 at the cone large end) and the second when the edge of the work piece extends beyond the cone outside surface (lower work piece bottom edge in fig 6 at the cone large end). These configurations can appear at either the top or the bottom of the diagram or the top and bottom can be the same configuration. Figure 8 illustrates these in greater detail.

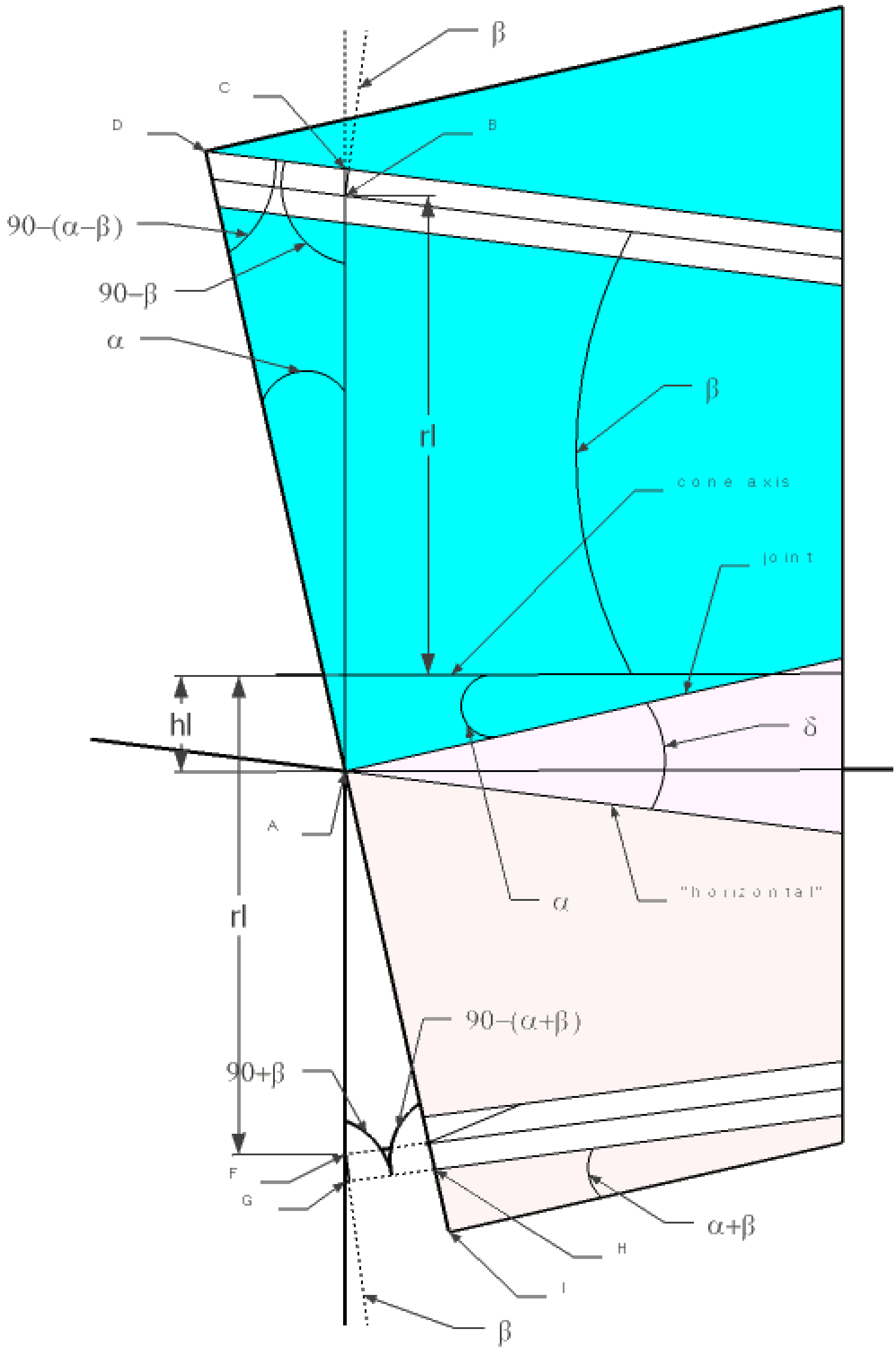


Figure 8 Expanded part of figure 6

The procedure to find wtu in fig. 8 is:-

extend the line AB to C using eq 48. Divide AC by sin(90-(α-β)) which now equals line AD divided by sin(90-β) (sine rule). AD is the minimum thickness of the upper work piece.

$$BC=\frac{wt/2}{\cos(\beta)} \qquad 48$$

Performing this procedure gives

$$wtu=\frac{((rl-hl)\cos(\beta)+wt/2)}{\cos(\alpha-\beta)} \qquad 49$$

The procedure to find wtl in fig. 8 is:-

Extend the line AF to G using eq 48.

Use line AG divided by sin(90-(α+β)) which now equals line AH divided by sin(90+β), to find the length AH. (sine rule). Add the length HI given by eq 50. AI is the minimum thickness of the lower work piece.

$$HI=wj*\sin(\alpha+\beta) \qquad 50$$

Performing this procedure gives

$$wtl=\frac{((rl+hl)\cos(\beta)+wt/2+wj*\sin(\alpha+\beta))}{\cos(\alpha+\beta)} \qquad 51$$

Notice that eq 49 is similar to eq 51with the addition of a third term inside the bracket in the numerator and the sign changes for hl and β.

That third term can also appear in the equation for wtu (when it will be wj*sin(α-β)) and it may not appear in the equation for wtl and indeed may not appear in either equation

The criteria for including that third term is: $\alpha<\beta\wedge|\alpha|>|\beta|$

To summarise:

The upper work piece thickness is given by eq 49 or eq 52 and the lower work piece thickness is given by eq 51 or equation 53.

$$wtu=\frac{((rh-hl)\cos(\beta)+wt/2+wj*\sin(\alpha-\beta))}{\cos(\alpha-\beta)} \qquad 52$$

$$wtl=\frac{(rh+hl)\cos(\beta)+wt/2}{\cos(\alpha+\beta)} \qquad 53$$

The symmetric case

This is the situation where both the upper work piece and the lower work piece are to be used in the same ring and we will assume there is no “dish”. Figure 9 is a cross section of such a design. Here the two work pieces are separated by a wedge of waste wood and the thickness of the two work pieces are the same.

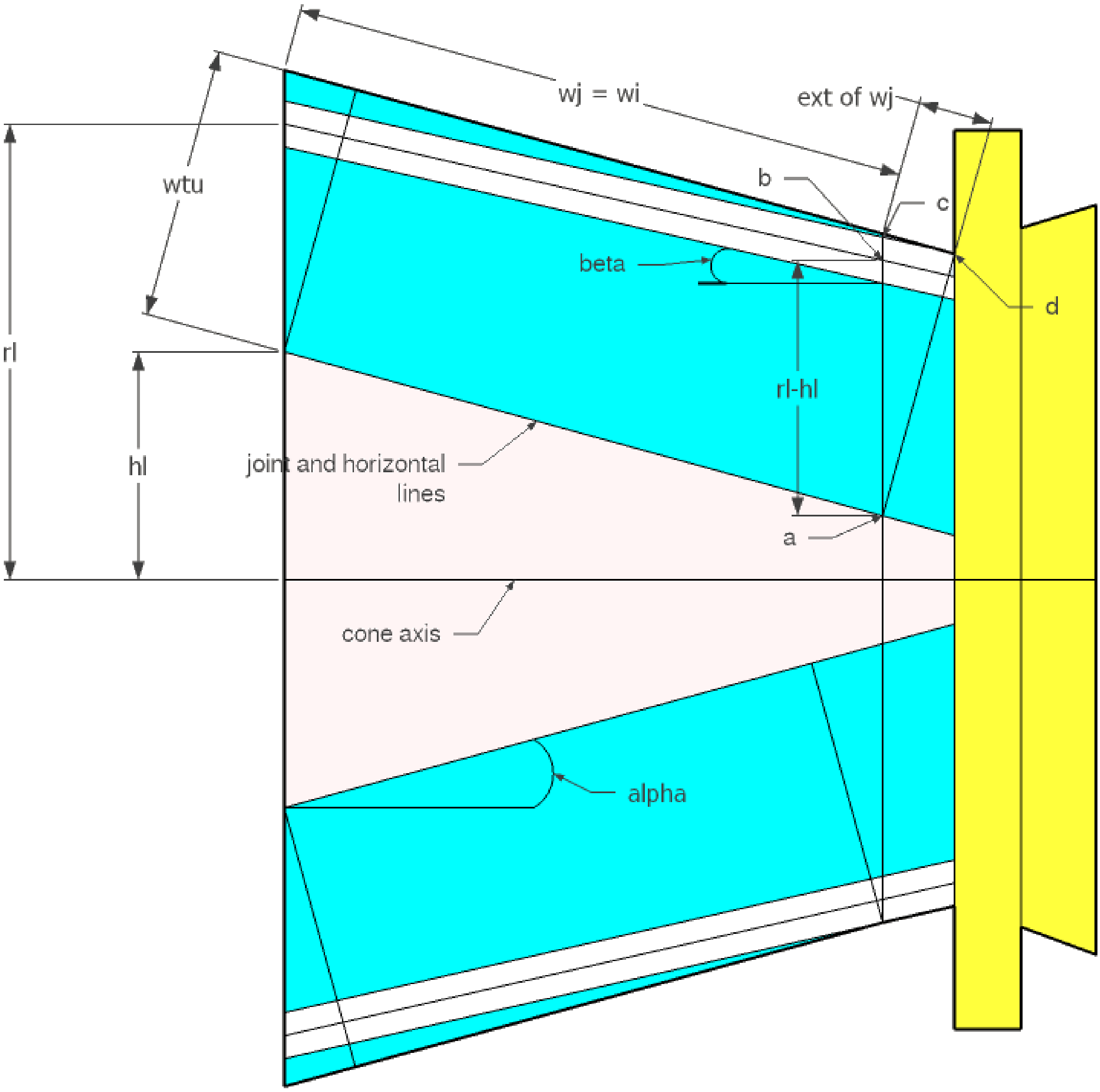


Figure 9 Illustration of a symmetric design

The work holder doesn't have an angled front face and to get vertical inner and outer surfaces on the ring it will be necessary to turn the end face of the large end of the cone, and to part off the smaller end, at right angles to the joint line. Doing this will shorten the cone and to compensate for this wj will need to be extended at the small end of the cone.

The procedure to determine the extension of wj is:

line ac divided by sin(90-(β-α)) equals line ad divided by sin(90+β) gives line ad.

Then the extension of wj is given by ad*sin(α).

Performing this procedure gives

$$wjx=\frac{((rl-hl)\cos(\beta)+wt/2)\tan(\alpha)}{\cos(\beta-\alpha)} \qquad 54$$

By a similar procedure for obtaining eq. 52 we get

$$wtu=wtl=\frac{\left((rl-hl)\cos(\beta)+\frac{wt}{2}+wjx*\sin(\beta-\alpha)\right)\cos(\alpha)}{\cos(\beta)} \qquad 55$$

The third term in the numerator is 0 if (β-α)<0

Length of the wedge is wjx*cos(α)

the angle on each side of the wedge is α and the width of the wedge at the outer end of the cone is 2*hl. Similarly the width of the wedge at the inner edge of the cone is 2*hs.

In the case that the “dish” is not zero then wj is replaced by wj/cos(δ).